

ON STABILIZATION OF SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS WITH A GRADIENT TERM

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ABSTRACT. For parabolic equations of the form

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial^2 u}{\partial x_i \partial x_j} + f(x, u, Du) = 0 \quad \text{in } \mathbb{R}_+^{n+1},$$

where $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$, $n \geq 1$, $D = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ is the gradient operator, and f is some function, we obtain conditions guaranteeing that every solution tends to zero as $t \rightarrow \infty$.

1. INTRODUCTION

We study solutions of the equations

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial^2 u}{\partial x_i \partial x_j} + f(x, u, Du) = 0 \quad \text{in } \mathbb{R}_+^{n+1}, \quad (1.1)$$

where $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$, $n \geq 1$, and $D = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ is the gradient operator. We assume that

$$\sum_{i,j=1}^n a_{ij}(x, \zeta) \xi_i \xi_j > 0$$

for all $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R} \setminus \{0\}$, and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$. Also let there are locally bounded measurable functions $g : (0, \infty) \rightarrow (0, \infty)$, $h : (0, \infty) \rightarrow (0, \infty)$, and $p : \mathbb{R}^n \rightarrow [0, \infty)$ such that

$$\inf_K g > 0 \quad \text{and} \quad \inf_K h > 0$$

for any compact set $K \subset (0, \infty)$ and, moreover,

$$f(x, \zeta, \lambda x) \operatorname{sign} \zeta \geq p(x) g(|\zeta|) \left(1 + \sum_{i,j=1}^n |a_{ij}(x, \zeta)| \right) \quad (1.2)$$

for all $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R} \setminus \{0\}$, and $0 \leq \lambda \leq p(x) h(|\zeta|)$, where

$$\operatorname{sign} \zeta = \begin{cases} 1, & \zeta > 0, \\ 0, & \zeta = 0, \\ -1, & \zeta < 0. \end{cases}$$

By a solution of (1.1) we mean a function u that has two continuous derivatives with respect to x and one continuous derivative with respect to t and satisfies equation (1.1) in the classical sense [5].

1991 *Mathematics Subject Classification.* 35J15, 35J60, 35J61, 35J62, 35J92.

Key words and phrases. Nonlinear parabolic equations, Unbounded domains, Stabilization to zero.

The research was supported by RFBR, grant 11-01-12018-ofi-m-2011.

No smoothness assumptions on a_{ij} and f are imposed, we do not even require these functions to be measurable.

Let us denote $B_r^x = \{y \in \mathbb{R}^n : |y - x| < r\}$, $S_r^x = \{y \in \mathbb{R}^n : |y - x| = r\}$, and $Q_{x,r}^{t_1,t_2} = B_r^x \times (t_1, t_2)$. In the case of $x = 0$, we write B_r , S_r , and $Q_r^{t_1,t_2}$ instead of B_r^0 , S_r^0 , and $Q_{0,r}^{t_1,t_2}$, respectively.

Put

$$q(r) = \inf_{B_r} p, \quad r \in (0, \infty).$$

For any function $\varphi : (0, \infty) \rightarrow \mathbb{R}$ and a real number $\theta > 1$ we also denote

$$\varphi_\theta(\zeta) = \inf_{(\zeta/\theta, \theta\zeta)} \varphi, \quad \zeta \in (0, \infty).$$

The questions treated in this paper were investigated earlier by a number of authors [1–7, 10, 11]. Below, we obtain conditions guaranteeing that every solution of (1.1) tends to zero as $t \rightarrow \infty$. These conditions take into account the dependence of the function f on the gradient term Du . Our results are applicable to a wide class of nonlinear equations (see Examples 2.1–2.4).

2. MAIN RESULTS

Theorem 2.1. *Let*

$$\int_1^\infty r q(r) dr = \infty \tag{2.1}$$

and, moreover,

$$\int_1^\infty (g_\theta(\zeta)\zeta)^{-1/2} d\zeta < \infty \tag{2.2}$$

and

$$\int_1^\infty \frac{d\zeta}{h_\theta(\zeta)} < \infty \tag{2.3}$$

for some real number $\theta > 1$. Then any solution of (1.1) stabilizes to zero uniformly on an arbitrary compact set $K \subset \mathbb{R}^n$ as $t \rightarrow \infty$, i.e.

$$\lim_{t \rightarrow \infty} \sup_{x \in K} |u(x, t)| = 0.$$

Theorem 2.1 will be proved later. Now, we demonstrate its application.

Example 2.1. Consider the equation

$$u_t = \Delta u + b(x, u, Du) - c(x)|u|^{\sigma-1}u \quad \text{in } \mathbb{R}_+^{n+1}, \tag{2.4}$$

where

$$|b(x, \zeta, \xi)| \leq b_0(1 + |x|)^k \zeta^\mu |\xi|^\alpha, \quad b_0 = \text{const} > 0, \tag{2.5}$$

and

$$c(x) \geq c_0(1 + |x|)^l, \quad c_0 = \text{const} > 0, \tag{2.6}$$

for all $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R} \setminus \{0\}$, and $\xi \in \mathbb{R}^n$. We assume that $\alpha > 0$ whereas μ , σ , k , and l can be arbitrary real numbers.

According to Theorem 2.1, if

$$\min\{l - k + \alpha, l + 2\} \geq 0 \tag{2.7}$$

and

$$\sigma > \max\{1, \alpha + \mu\}, \tag{2.8}$$

then any solution of (2.4) stabilizes to zero uniformly on an arbitrary compact subset of \mathbb{R}^n as $t \rightarrow \infty$. Really, taking $f(x, \zeta, \xi) = c(x)|\zeta|^{\sigma-1}\zeta - b(x, \zeta, \xi)$, $g(\zeta) = \zeta^\sigma$, and $h(\zeta) = \zeta^{(\sigma-\mu)/\alpha}$, we obtain that (1.2) is valid with

$$p(x) = p_0(1 + |x|)^{\min\{(l-k)/\alpha-1, l\}}, \quad (2.9)$$

where $p_0 > 0$ is a sufficiently small real number. In so doing, relation (2.1) is equivalent to (2.7) while (2.2) and (2.3) are equivalent to (2.8).

At the same time, if (2.7) is not fulfilled, then for some functions b and c satisfying (2.5) and (2.6) equation (2.4) has a positive solution which does not depend on the variable t . This solution obviously can not stabilize to zero as $t \rightarrow \infty$. In turn, if (2.8) is not fulfilled, then for all real numbers k and l there exist functions b and c such that (2.5) and (2.6) are valid and equation (2.4) has a positive solution independent of t . In this sense, conditions (2.7) and (2.8) are the best possible.

We also note that (2.7) and (2.8) correspond to the blow-up conditions for non-negative solutions of the elliptic inequalities

$$\Delta u + b(x, u, Du) \geq c(x)u^\sigma \quad \text{in } \mathbb{R}^n$$

considered in [8, Example 2.1] for $\mu = 0$ and $1 \leq \alpha \leq 2$.

Example 2.2. In (2.4), let the functions b and c satisfy the relations

$$|b(x, \zeta, \xi)| \leq b_0(1 + |x|)^k \log^s(2 + |x|)\zeta^\mu|\xi|^\alpha, \quad b_0 = \text{const} > 0, \quad (2.10)$$

and

$$c(x) \geq c_0(1 + |x|)^l \log^m(2 + |x|), \quad c_0 = \text{const} > 0, \quad (2.11)$$

for all $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R} \setminus \{0\}$, and $\xi \in \mathbb{R}^n$, where α , k , s , l , and m are real numbers with $\alpha > 0$ and

$$\min\{l - k + \alpha, l + 2\} = 0.$$

In other words, we consider the case of the critical exponents l and k in (2.7).

As in the previous example, we take $f(x, \zeta, \xi) = c(x)|\zeta|^{\sigma-1}\zeta - b(x, \zeta, \xi)$, $g(\zeta) = \zeta^\sigma$, and $h(\zeta) = \zeta^{(\sigma-\mu)/\alpha}$. It can be verified that (1.2) is valid for

$$p(x) = p_0(1 + |x|)^{-2} \log^\gamma(2 + |x|),$$

where $p_0 > 0$ is a sufficiently small real number and

$$\gamma = \begin{cases} m, & l + 2 < l - k + \alpha, \\ \min\{(m - s)/\alpha, m\}, & l + 2 = l - k + \alpha, \\ (m - s)/\alpha, & l + 2 > l - k + \alpha. \end{cases}$$

Thus, by Theorem 2.1, if (2.8) holds and

$$\gamma \geq -1, \quad (2.12)$$

then any solution of (2.4) stabilizes to zero uniformly on an arbitrary compact subset of \mathbb{R}^n as $t \rightarrow \infty$.

If (2.12) is not fulfilled and $n \geq 3$, then there exist functions b and c such that (2.10) and (2.11) are valid and (2.4) has a positive solution independent of t . Condition (2.8) is also the best possible.

Example 2.3. Consider the equation

$$u_t = \Delta u + b(x, u, Du) - c(x)|u|^{\sigma-1}u \log^\nu(1 + |u|) \quad \text{in } \mathbb{R}_+^{n+1}, \quad (2.13)$$

where the functions b and c satisfy (2.5) and (2.6) with $\alpha > 0$ and

$$\sigma = \max\{1, \alpha + \mu\},$$

i.e. we are interested in the case of the critical exponent σ in (2.8).

Taking $f(x, \zeta, \xi) = c(x)|\zeta|^{\sigma-1}\zeta \log^\nu(1+\zeta) - b(x, \zeta, \xi)$, $g(\zeta) = \zeta^\sigma \log^\nu(1+\zeta)$, and $h(\zeta) = \zeta^{(\sigma-\mu)/\alpha} \log^{\nu/\alpha}(1+\zeta)$, one can see that (1.2) holds with some function p of the form (2.9). Thus, in accordance with Theorem 2.1 if (2.7) is valid and

$$\nu > \begin{cases} 2, & \alpha + \mu < 1, \\ \max\{2, \alpha\}, & \alpha + \mu = 1, \\ \alpha, & \alpha + \mu > 1, \end{cases} \quad (2.14)$$

then any solution of (2.13) stabilizes to zero uniformly on an arbitrary compact subset of \mathbb{R}^n as $t \rightarrow \infty$.

If (2.7) is not fulfilled, then there are functions b and c such that (2.5) and (2.6) hold and equation (2.13) has a positive solution independent of the variable t . In turn, if (2.14) is not fulfilled, then for all real numbers k and l there exist functions b and c satisfying (2.5) and (2.6) for which (2.13) has a positive solution independent of t .

Example 2.4. In the equation

$$u_t = \Delta u + b(x, Du) - c(x, u) \quad \text{in } \mathbb{R}_+^{n+1}, \quad (2.15)$$

let

$$|b(x, \xi)| \leq \varphi(|\xi|) \quad (2.16)$$

and

$$c(x, \zeta) \operatorname{sign} \zeta \geq \psi(|\zeta|) \quad (2.17)$$

for all $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R} \setminus \{0\}$, and $\xi \in \mathbb{R}^n$, where $\varphi : [0, \infty) \rightarrow [0, \infty)$ and $\psi : (0, \infty) \rightarrow (0, \infty)$ are non-decreasing continuous functions. We also assume that $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a bijection and

$$\liminf_{\zeta \rightarrow \infty} \frac{\varphi^{-1}(\zeta)}{\varphi^{-1}(2\zeta)} > 0,$$

where φ^{-1} is the inverse function of φ .

According to Theorem 2.1, if

$$\int_1^\infty (\psi(\zeta)\zeta)^{-1/2} d\zeta < \infty \quad (2.18)$$

and

$$\int_1^\infty \frac{d\zeta}{\varphi^{-1} \circ \psi(\zeta)} < \infty, \quad (2.19)$$

then any solution of (2.15) stabilizes to zero uniformly on an arbitrary compact subset of \mathbb{R}^n as $t \rightarrow \infty$. Indeed, we put $f(x, \zeta, \xi) = c(x, \zeta) - b(x, \xi)$. It does not present any particular problem to verify that (1.2) is valid with $g(\zeta) = \psi(\zeta)$, $h(\zeta) = \varphi^{-1}(\varepsilon\psi(\zeta))$, and $p(x) = \varepsilon$, where $\varepsilon > 0$ is a sufficiently small real number. In so doing, condition (2.1) is certainly satisfied while (2.2) and (2.3) are equivalent to (2.18) and (2.19), respectively.

We can show that, in the case where at least one of conditions (2.18), (2.19) is not fulfilled, there are functions b and c such that (2.16) and (2.17) hold and equation (2.15) has a positive solution which does not depend on t .

Proof of Theorem 2.1 relies on the following assertion.

Theorem 2.2. *Suppose that u is a solution of the inequality*

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial^2 u}{\partial x_i \partial x_j} + f(x, u, Du) \leq 0 \quad \text{in } \mathbb{R}_+^{n+1}. \quad (2.20)$$

Also let there exist a real number $\theta > 1$ such that (2.1), (2.2), and (2.3) hold. Then

$$\lim_{t \rightarrow \infty} \sup_{x \in K} u_+(x, t) = 0$$

for any compact set $K \subset \mathbb{R}^n$, where

$$u_+(x, t) = \begin{cases} u(x, t), & u(x, t) > 0, \\ 0, & u(x, t) \leq 0. \end{cases}$$

Proof of Theorem 2.2 is given in Section 3. As for equation (1.1), solutions of (2.20) are understood in the classical sense.

Proof of Theorem 2.1. We apply Theorem 2.2 to the functions u and $-u$. \square

3. PROOF OF THEOREM 2.2

In this section we assume that hypotheses of Theorem 2.2 are satisfied. Let us denote

$$L = \sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial^2}{\partial x_i \partial x_j}.$$

By C we mean various positive constants which can depend only on n and θ .

Definition 3.1 ([5]). A point (x, t) belongs to the upper lid $\gamma(\Omega)$ of an open set $\Omega \subset \mathbb{R}_+^{n+1}$ if there exist real numbers $r > 0$ and $\varepsilon > 0$ such that $Q_{x,r}^{t-\varepsilon, t} \subset \Omega$ and $Q_{x,r}^{t, t+\varepsilon} \cap \Omega = \emptyset$. The set $\Gamma(\Omega) = \partial\Omega \setminus \gamma(\Omega)$ is called the proper (or parabolic) boundary of Ω .

Lemma 3.1. *Let v and w satisfy the inequalities*

$$Lv - v_t - f(x, u, Dv) \geq Lw - w_t - f(x, u, Dw) \quad \text{in } \omega \cup \gamma(\omega) \quad (3.1)$$

and

$$v \leq w \quad \text{on } \Gamma(\omega), \quad (3.2)$$

where $\omega \neq \emptyset$ is a bounded open subset of \mathbb{R}_+^n such that u is a positive function on $\omega \cup \gamma(\omega)$. Then

$$v \leq w \quad \text{on } \omega \cup \gamma(\omega). \quad (3.3)$$

Proof. Lemma 3.1 is the standard maximum principle for parabolic inequalities in bounded domains [5]. The only subtlety is that (3.1) contains the function f . However, this fact can not affect the proof in a significant way. Not wanting to be unfounded, we give this proof in detail.

It can obviously be assumed that, in formula (3.1), the inequality is strong; otherwise we replace v by $v - \varepsilon t$ and pass to the limit as $\varepsilon \rightarrow +0$.

Denote

$$\varphi = v - w.$$

If (3.3) is not valid, then there exists a real number $\mu > 0$ for which the set $\omega_\mu = \{(x, t) \in \omega : \varphi(x, t) > \mu\}$ is not empty.

According to (3.2), the closure $\bar{\omega}_\mu$ of the set ω_μ is contained in $\omega \cup \gamma(\omega)$. Let us take a point $(x', t') \in \bar{\omega}_\mu$ such that

$$\varphi(x', t') = \sup_{\bar{\omega}_\mu} \varphi. \quad (3.4)$$

We have $D\varphi(x', t') = 0$ and $\varphi_t(x', t') \geq 0$ or, in other words, $Dv(x', t') = Dw(x', t')$ and $v_t(x', t') \geq w_t(x', t')$. It can easily be seen that

$$\sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \Big|_{x=x', t=t'} \leq 0; \quad (3.5)$$

otherwise, introducing the new coordinates $y = y(x)$ such that

$$\sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \Big|_{x=x', t=t'} = \sum_{i=1}^n \frac{\partial^2 \varphi}{\partial y_i^2} \Big|_{y=y(x'), t=t'},$$

we obtain

$$\frac{\partial^2 \varphi}{\partial y_i^2} \Big|_{y=y', t=t'} > 0$$

for some $1 \leq i \leq n$. This contradicts (3.4).

At the same time, (3.5) is equivalent to the relation

$$\sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial^2 v}{\partial x_i \partial x_j} \Big|_{x=x', t=t'} \leq \sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial^2 w}{\partial x_i \partial x_j} \Big|_{x=x', t=t'};$$

therefore, we arrive at a contradiction with our assumption that the inequality in (3.1) is strong.

The proof is completed. \square

Corollary 3.1. *Let v satisfy the inequality*

$$Lv - v_t - f(x, u, Dv) \geq 0 \quad \text{in } \omega \cup \gamma(\omega),$$

where $\omega \neq \emptyset$ is a bounded open subset of \mathbb{R}_+^n such that u is a positive function on $\omega \cup \gamma(\omega)$. Then

$$v(x, t) \leq \sup_{\Gamma(\omega)} v$$

for all $(x, t) \in \omega \cup \gamma(\omega)$.

Proof. In Lemma 3.1, we take

$$w = \sup_{\Gamma(\omega)} v.$$

\square

Lemma 3.2. *Let $0 < r_1 < r_2$ and $0 < \tau_1 < \tau_2 < \tau$ be real numbers with*

$$\sup_{Q_{r_1}^{\tau-\tau_1, \tau}} u > 0.$$

Then at least one of the following three estimates is valid:

$$\begin{aligned} M_2 - M_1 &\geq C(r_2 - r_1)^2 q(r_2) \lim_{\mu \rightarrow M_1 - 0} \inf_{[\mu, M_2]} g, \\ M_2 - M_1 &\geq C(r_2 - r_1) r_2 q(r_2) \lim_{\mu \rightarrow M_1 - 0} \inf_{[\mu, M_2]} h, \\ M_2 - M_1 &\geq C(\tau_2 - \tau_1) q(r_2) \lim_{\mu \rightarrow M_1 - 0} \inf_{[\mu, M_2]} g, \end{aligned}$$

where

$$M_1 = \sup_{Q_{r_1}^{\tau-\tau_1, \tau}} u \quad \text{and} \quad M_2 = \sup_{Q_{r_2}^{\tau-\tau_2, \tau}} u.$$

Proof. Assume that μ is a real number satisfying the condition $0 < \mu < M_1$. Also let $r_0 = (r_1 + r_2)/2$ and $\varphi \in C^\infty(\mathbb{R})$ be a non-decreasing function such that

$$\varphi|_{(-\infty, 0]} = 0 \quad \text{and} \quad \varphi|_{[1, \infty)} = 1.$$

We denote

$$w(x, t) = k_1 \varphi \left(\frac{|x| - r_0}{r_2 - r_0} \right) + k_2 \varphi \left(\frac{\tau - \tau_1 - t}{\tau_2 - \tau_1} \right),$$

where

$$k_1 = \min \left\{ \frac{(r_2 - r_0)^2 q(r_2) \inf_{[\mu, M_2]} g}{2 \|\varphi\|_{C^2([0, 1])}}, \frac{(r_2 - r_0) r_2 q(r_2) \inf_{[\mu, M_2]} h}{2 \|\varphi\|_{C^1([0, 1])}} \right\}$$

and

$$k_2 = \frac{(\tau_2 - \tau_1) q(r_2) \inf_{[\mu, M_2]} g}{\|\varphi\|_{C^1([0, 1])}}.$$

Further, let $\Omega = \{(x, t) \in \mathbb{R}_+^n : u(x, t) > \mu\}$ and $\omega = \Omega \cap Q_{r_2}^{\tau-\tau_2, \tau}$. By direct calculation, it can be shown that

$$\begin{aligned} Lw - w_t &= k_1 \varphi'' \left(\frac{|x| - r_0}{r_2 - r_0} \right) \sum_{i,j=1}^n \frac{a_{ij}(x, u) x_i x_j}{|x|^2 (r_2 - r_0)^2} + k_1 \varphi' \left(\frac{|x| - r_0}{r_2 - r_0} \right) \sum_{i=1}^n \frac{a_{ii}(x, u)}{|x| (r_2 - r_0)} \\ &\quad - k_1 \varphi' \left(\frac{|x| - r_0}{r_2 - r_0} \right) \sum_{i,j=1}^n \frac{a_{ij}(x, u) x_i x_j}{|x|^3 (r_2 - r_0)} + \frac{k_2}{\tau_2 - \tau_1} \varphi' \left(\frac{\tau - \tau_1 - t}{\tau_2 - \tau_1} \right) \\ &\leq p(|x|) g(u) \left(1 + \sum_{i,j=1}^n |a_{ij}(x, u)| \right) \end{aligned} \quad (3.6)$$

for all $(x, t) \in \omega \cup \gamma(\omega)$. In so doing, we obviously have

$$Dw = \frac{k_1 x}{|x| (r_2 - r_0)} \varphi' \left(\frac{|x| - r_0}{r_2 - r_0} \right)$$

and

$$0 \leq \frac{k_1}{|x| (r_2 - r_0)} \varphi' \left(\frac{|x| - r_0}{r_2 - r_0} \right) \leq p(|x|) h(u)$$

for all $(x, t) \in \omega \cup \gamma(\omega)$, whence in accordance with (1.2) it follows that

$$f(x, u, Dw) \geq p(|x|) g(u) \left(1 + \sum_{i,j=1}^n |a_{ij}(x, u)| \right)$$

for all $(x, t) \in \omega \cup \gamma(\omega)$. Combining the last inequality with (3.6), we obtain

$$Lw - w_t \leq f(x, u, Dw) \quad \text{in } \omega \cup \gamma(\omega). \quad (3.7)$$

Let us show that

$$\sup_{\overline{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} (u - w) \geq \sup_{\omega} (u - w). \quad (3.8)$$

Really, if (3.8) is not valid, then

$$\sup_{\overline{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} (u - w) < \sup_{\omega} (u - w) - \varepsilon \quad (3.9)$$

for some $\varepsilon > 0$. Without loss of generality, it can also be assumed that $\mu < M_1 - \varepsilon$. We denote

$$v = u - \sup_{\omega} (u - w) + \varepsilon. \quad (3.10)$$

From (2.20), it follows that

$$Lv - v_t \geq f(x, t, u, Dv) \quad \text{in } \omega \cup \gamma(\omega).$$

Combining this with (3.7), we immediately obtain (3.1). Let us now establish the validity of inequality (3.2). We have

$$\Gamma(\omega) \subset \left(\Gamma(\Omega) \cap \overline{Q_{r_2}^{\tau-\tau_2, \tau}} \right) \cup (\bar{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})) \quad (3.11)$$

and

$$u|_{\Gamma(\Omega) \cap \overline{Q_{r_2}^{\tau-\tau_2, \tau}}} = \mu. \quad (3.12)$$

Relations (3.10) and (3.12) imply that

$$v|_{\Gamma(\Omega) \cap \overline{Q_{r_2}^{\tau-\tau_2, \tau}}} = \mu - \sup_{\omega} (u - w) + \varepsilon.$$

In so doing,

$$\sup_{\omega} (u - w) \geq \sup_{\Omega \cap Q_{r_1}^{\tau-\tau_1, \tau}} u = M_1$$

since $\Omega \cap Q_{r_1}^{\tau-\tau_1, \tau} \subset \omega$ and w is equal to zero on $Q_{r_1}^{\tau-\tau_1, \tau}$; therefore, we obtain

$$v|_{\Gamma(\Omega) \cap \overline{Q_{r_2}^{\tau-\tau_2, \tau}}} \leq \mu - M_1 + \varepsilon < 0.$$

The last formula and the fact that w is a non-negative function yield

$$v \leq w \quad \text{on } \Gamma(\Omega) \cap \overline{Q_{r_2}^{\tau-\tau_2, \tau}}.$$

At the same time, taking into account (3.9) and (3.10), we have

$$\sup_{\bar{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} (v - w) = \sup_{\bar{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} (u - w) - \sup_{\omega} (u - w) + \varepsilon < 0.$$

Consequently, one can assert that (3.2) is fulfilled. Thus, by Lemma 3.1, inequality (3.3) holds or, in other words,

$$u - \sup_{\omega} (u - w) + \varepsilon \leq w \quad \text{on } \omega \cup \gamma(\omega),$$

whence it follows that

$$\sup_{\omega \cup \gamma(\omega)} (u - w) + \varepsilon \leq \sup_{\omega} (u - w).$$

This contradiction proves (3.8).

Since w is equal to zero on $Q_{r_1}^{\tau-\tau_1, \tau}$, formula (3.8) implies the estimate

$$\sup_{\bar{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} (u - w) \geq \sup_{\omega \cap Q_{r_1}^{\tau-\tau_1, \tau}} u$$

from which, by the relations

$$\sup_{\bar{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} u - \inf_{\bar{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} w \geq \sup_{\bar{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} (u - w)$$

and

$$\sup_{\omega \cap Q_{r_1}^{\tau-\tau_1, \tau}} u = M_1,$$

we obtain

$$\sup_{\overline{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} u - \inf_{\overline{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} w \geq M_1. \quad (3.13)$$

From Corollary 3.1, it follows that

$$\sup_{\Gamma(\omega)} u = \sup_{\overline{\omega}} u = M_2.$$

In addition, (3.12) implies the inequality

$$u|_{\Gamma(\Omega) \cap \overline{Q_{r_2}^{\tau-\tau_2, \tau}}} < M_2;$$

therefore, inclusion (3.11) allows us to assert that

$$\sup_{\overline{\omega} \cap \Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} u = \sup_{\Gamma(\omega)} u = M_2.$$

Combining this with (3.13), we obtain

$$M_2 - M_1 \geq \inf_{\Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} w.$$

To complete the proof, it remains to note that

$$\inf_{\Gamma(Q_{r_2}^{\tau-\tau_2, \tau})} w \geq \min\{k_1, k_2\}.$$

□

Lemma 3.3. *Let $r > 0$ and $t > 0$ be real numbers such that $4r^2 < t$ and*

$$\sup_{Q_{2r}^{t-4r^2, t}} u \geq \theta^{1/2} \sup_{Q_r^{t-r^2, t}} u > 0. \quad (3.14)$$

Then at least one of the following two estimates is valid:

$$\int_{M_1}^{M_2} (g_\theta(\zeta)\zeta)^{-1/2} d\zeta \geq C \int_r^{2r} q^{1/2}(2\rho) d\rho, \quad (3.15)$$

$$\int_{M_1}^{M_2} \frac{d\zeta}{h_\theta(\zeta)} \geq C \int_r^{2r} \rho q(2\rho) d\rho, \quad (3.16)$$

where

$$M_1 = \sup_{Q_r^{t-r^2, t}} u \quad \text{and} \quad M_2 = \sup_{Q_{2r}^{t-4r^2, t}} u. \quad (3.17)$$

Proof. Let k be the maximal positive integer for which $\theta^{k/2} M_1 \leq M_2$. We put $m_i = \theta^{i/2} M_1$, $i = 0, \dots, k-1$, and $m_k = M_2$. It can easily be seen that

$$\theta^{1/2} m_i \leq m_{i+1} < \theta m_i, \quad i = 0, \dots, k-1.$$

Let us further take an increasing sequence of real numbers $\{r_i\}_{i=0}^k$ such that $r_0 = r$, $r_k = 2r$, and

$$\sup_{Q_{r_i}^{t-r_i^2, t}} u = m_i, \quad i = 1, \dots, k-1.$$

Since u is a continuous function in \mathbb{R}_+^n , such a sequence obviously exists.

By Lemma 3.2, for any $i \in \{0, \dots, k-1\}$ at least one of the following three inequalities is valid:

$$m_{i+1} - m_i \geq C(r_{i+1} - r_i)^2 q(r_{i+1}) \lim_{\mu \rightarrow m_i - 0} \inf_{[\mu, m_{i+1}]} g, \quad (3.18)$$

$$m_{i+1} - m_i \geq C(r_{i+1} - r_i)r_{i+1}q(r_{i+1}) \lim_{\mu \rightarrow m_i - 0} \inf_{[\mu, m_{i+1}]} h, \quad (3.19)$$

$$m_{i+1} - m_i \geq C(r_{i+1}^2 - r_i^2)q(r_{i+1}) \lim_{\mu \rightarrow m_i - 0} \inf_{[\mu, m_{i+1}]} g. \quad (3.20)$$

If (3.18) is valid, then we have

$$\left(\frac{m_{i+1} - m_i}{\lim_{\mu \rightarrow m_i - 0} \inf_{[\mu, m_{i+1}]} g} \right)^{1/2} \geq C(r_{i+1} - r_i)q^{1/2}(r_{i+1}).$$

Since

$$\int_{m_i}^{m_{i+1}} (g_\theta(\zeta)\zeta)^{-1/2} d\zeta \geq C \left(\frac{m_{i+1} - m_i}{\lim_{\mu \rightarrow m_i - 0} \inf_{[\mu, m_{i+1}]} g} \right)^{1/2},$$

this implies the estimate

$$\int_{m_i}^{m_{i+1}} (g_\theta(\zeta)\zeta)^{-1/2} d\zeta \geq C(r_{i+1} - r_i)q^{1/2}(r_{i+1}). \quad (3.21)$$

In turn, if (3.19) holds, then

$$\frac{m_{i+1} - m_i}{\lim_{\mu \rightarrow m_i - 0} \inf_{[\mu, m_{i+1}]} h} \geq C(r_{i+1} - r_i)r_{i+1}q(r_{i+1}),$$

whence in accordance with the inequality

$$\int_{m_i}^{m_{i+1}} \frac{d\zeta}{h_\theta(\zeta)} \geq \frac{C(m_{i+1} - m_i)}{\lim_{\mu \rightarrow m_i - 0} \inf_{[\mu, m_{i+1}]} h}$$

we obtain

$$\int_{m_i}^{m_{i+1}} \frac{d\zeta}{h_\theta(\zeta)} \geq C(r_{i+1} - r_i)r_{i+1}q(r_{i+1}). \quad (3.22)$$

Let us also note that (3.20) implies (3.18); therefore, in this case, we again arrive at (3.21). Thus, for any $i \in \{0, \dots, k-1\}$ at least one of estimates (3.21), (3.22) is valid. We denote by Ξ_1 the set of integers $i \in \{0, \dots, k-1\}$ for which (3.21) holds. In so doing, let $\Xi_2 = \{0, \dots, k-1\} \setminus \Xi_1$.

At first assume that

$$\sum_{i \in \Xi_1} (r_{i+1} - r_i) \geq \frac{r_k - r_0}{2}. \quad (3.23)$$

Then, summing (3.21) over all $i \in \Xi_1$, we have

$$\int_{M_1}^{M_2} (g_\theta(\zeta)\zeta)^{-1/2} d\zeta \geq C(r_k - r_0)q^{1/2}(r_k).$$

This implies (3.15).

Now, let (3.23) is not valid. Then

$$\sum_{i \in \Xi_2} (r_{i+1} - r_i) \geq \frac{r_k - r_0}{2};$$

therefore, summing (3.22) over all $i \in \Xi_2$, we conclude that

$$\int_{M_1}^{M_2} \frac{d\zeta}{h_\theta(\zeta)} \geq C(r_k - r_0)r_0q(r_k),$$

whence (3.16) immediately follows.

The proof is completed. \square

Lemma 3.4. *In the hypotheses of Lemma 3.3, let the inequality*

$$\theta^{1/2} \sup_{Q_r^{t-r^2,t}} u \geq \sup_{Q_{2r}^{t-4r^2,t}} u > 0$$

be fulfilled instead of (3.14). Then at least one of the following two estimates is valid:

$$\int_{M_1}^{M_2} \frac{d\zeta}{g_{\sqrt{\theta}}(\zeta)} \geq C \int_r^{2r} \rho q(2\rho) d\rho, \quad (3.24)$$

$$\int_{M_1}^{M_2} \frac{d\zeta}{h_{\sqrt{\theta}}(\zeta)} \geq C \int_r^{2r} \rho q(2\rho) d\rho, \quad (3.25)$$

where M_1 and M_2 are defined by (3.17).

Proof. By Lemma 3.2, we obviously obtain either

$$M_2 - M_1 \geq Cr^2 q(2r) \lim_{\mu \rightarrow M_1 - 0} \inf_{[\mu, M_2]} g \quad (3.26)$$

or

$$M_2 - M_1 \geq Cr^2 q(2r) \lim_{\mu \rightarrow M_1 - 0} \inf_{[\mu, M_2]} h. \quad (3.27)$$

If (3.26) holds, then

$$\frac{M_2 - M_1}{\lim_{\mu \rightarrow M_1 - 0} \inf_{[\mu, M_2]} g} \geq Cr^2 q(2r).$$

Thus, taking into account the inequality

$$\int_{M_1}^{M_2} \frac{d\zeta}{g_{\sqrt{\theta}}(\zeta)} \geq C \frac{M_2 - M_1}{\lim_{\mu \rightarrow M_1 - 0} \inf_{[\mu, M_2]} g},$$

we can assert that

$$\int_{M_1}^{M_2} \frac{d\zeta}{g_{\sqrt{\theta}}(\zeta)} \geq Cr^2 q(2r),$$

whence (3.24) follows at once.

Analogously, (3.27) implies the estimate

$$\int_{M_1}^{M_2} \frac{d\zeta}{h_{\sqrt{\theta}}(\zeta)} \geq Cr^2 q(2r)$$

from which (3.25) can be obtained.

The proof is completed. \square

Lemma 3.5. *Let $0 < \alpha \leq 1$, $\sigma > 1$, $\nu > 1$, $M_1 > 0$, and $M_2 > 0$ be real numbers with $M_2 \geq \nu M_1$. Then*

$$\left(\int_{M_1}^{M_2} \psi_{\sigma}^{-\alpha}(s) s^{\alpha-1} ds \right)^{1/\alpha} \geq A \int_{M_1}^{M_2} \frac{ds}{\psi(s)}$$

for any measurable function $\psi : (0, \infty) \rightarrow (0, \infty)$ such that $\psi_{\sigma}(s) > 0$ for all $s \in (0, \infty)$, where $A > 0$ is a constant depending only on α , ν , and σ .

Lemma 3.6. *Let $0 < \alpha \leq 1$, $\sigma > 1$, $\nu > 1$, $r_1 > 0$, and $r_2 > 0$ be real numbers with $r_2 \geq \nu r_1$. Then*

$$\left(\int_{r_1}^{r_2} \varphi^{\alpha}(r) dr \right)^{1/\alpha} \geq A \int_{r_1}^{r_2} r^{1/\alpha-1} \varphi_{\sigma}(r) dr$$

for any measurable function $\varphi : [r_1, r_2] \rightarrow [0, \infty)$, where $A > 0$ is a constant depending only on α , ν , and σ .

Lemmas 3.5 and 3.6 are proved in [9, Lemmas 2.3 and 2.6].

Proof of Theorem 2.2. Let K be a compact subset of \mathbb{R}^n and, moreover, $r > 0$ and $t > 0$ be real numbers such that $K \subset B_r$ and $t > r^2$. If

$$\sup_{x \in K} u(x, t) \leq 0,$$

then

$$\sup_{x \in K} u_+(x, t) = 0;$$

therefore, we can assume that

$$\sup_{x \in K} u(x, t) > 0.$$

Let us take the maximal integer k satisfying the condition $4^k r^2 < t$. Also put $r_i = 2^i r$ and

$$m_i = \sup_{Q_{r_i}^{t-r_i^2, t}} u, \quad i = 0, \dots, k.$$

We show that

$$\left(\int_{m_0}^{\infty} (g_{\theta}(\zeta) \zeta)^{-1/2} d\zeta \right)^2 + \int_{m_0}^{\infty} \frac{d\zeta}{h_{\theta}(\zeta)} \geq C \int_{r_0}^{r_k} \rho q(4\rho) d\rho. \quad (3.28)$$

Really, by Lemmas 3.3 and 3.4, for any $i \in \{0, \dots, k-1\}$ at least one of the following three inequalities is valid:

$$\int_{m_i}^{m_{i+1}} (g_{\theta}(\zeta) \zeta)^{-1/2} d\zeta \geq C \int_{r_i}^{r_{i+1}} q^{1/2}(2\rho) d\rho, \quad (3.29)$$

$$\int_{m_i}^{m_{i+1}} \frac{d\zeta}{h_{\theta}(\zeta)} \geq C \int_{r_i}^{r_{i+1}} \rho q(2\rho) d\rho, \quad (3.30)$$

$$\int_{m_i}^{m_{i+1}} \frac{d\zeta}{g_{\sqrt{\theta}}(\zeta)} \geq C \int_{r_i}^{r_{i+1}} \rho q(2\rho) d\rho. \quad (3.31)$$

By Ξ_1 , Ξ_2 , and Ξ_3 we denote the sets of integers $i \in \{0, \dots, k-1\}$ satisfying relations (3.29), (3.30), and (3.31), respectively.

At first, let

$$\sum_{i \in \Xi_1} \int_{r_i}^{r_{i+1}} \rho q(4\rho) d\rho \geq \frac{1}{2} \int_{r_0}^{r_k} \rho q(4\rho) d\rho. \quad (3.32)$$

Summing (3.29) over all $i \in \Xi_1$, we obtain

$$\int_{m_0}^{\infty} (g_{\theta}(\zeta) \zeta)^{-1/2} d\zeta \geq C \sum_{i \in \Xi_1} \int_{r_i}^{r_{i+1}} q^{1/2}(2\rho) d\rho.$$

By Lemma 3.6, this implies the estimate

$$\left(\int_{m_0}^{\infty} (g_{\theta}(\zeta) \zeta)^{-1/2} d\zeta \right)^2 \geq C \sum_{i \in \Xi_1} \left(\int_{r_i}^{r_{i+1}} q^{1/2}(2\rho) d\rho \right)^2 \geq C \sum_{i \in \Xi_1} \int_{r_i}^{r_{i+1}} \rho q(4\rho) d\rho$$

from which (3.28) immediately follows.

Now, assume that (3.32) is not valid. Then

$$\sum_{i \in \Xi_2 \cup \Xi_3} \int_{r_i}^{r_{i+1}} \rho q(4\rho) d\rho \geq \frac{1}{2} \int_{r_0}^{r_k} \rho q(4\rho) d\rho. \quad (3.33)$$

In this case, summing (3.30) over all $i \in \Xi_2$, we have

$$\int_{m_0}^{\infty} \frac{d\zeta}{h_{\theta}(\zeta)} \geq C \sum_{i \in \Xi_2} \int_{r_i}^{r_{i+1}} \rho q(2\rho) d\rho. \quad (3.34)$$

Analogously, (3.31) allows us to assert that

$$\int_{m_0}^{\infty} \frac{d\zeta}{g_{\sqrt{\theta}}(\zeta)} \geq C \sum_{i \in \Xi_3} \int_{r_i}^{r_{i+1}} \rho q(2\rho) d\rho.$$

Since

$$\left(\int_{m_0}^{\infty} (g_{\theta}(\zeta)\zeta)^{-1/2} d\zeta \right)^2 \geq C \int_{m_0}^{\infty} \frac{d\zeta}{g_{\sqrt{\theta}}(\zeta)}$$

in view of Lemma 3.5, this yields the inequality

$$\left(\int_{m_0}^{\infty} (g_{\theta}(\zeta)\zeta)^{-1/2} d\zeta \right)^2 \geq C \sum_{i \in \Xi_3} \int_{r_i}^{r_{i+1}} \rho q(2\rho) d\rho. \quad (3.35)$$

Combining (3.33), (3.34), and (3.35), we again arrive at (3.28).

Further, assuming that r is fixed, we obviously obtain $r_k \rightarrow \infty$ as $t \rightarrow \infty$; therefore, in accordance with (2.1), (2.2), and (2.3) formula (3.28) implies that $m_0 \rightarrow 0$ as $t \rightarrow \infty$. Thus,

$$\sup_{Q_r^{t-r^2, t}} u_+ \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

The proof is completed. □

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